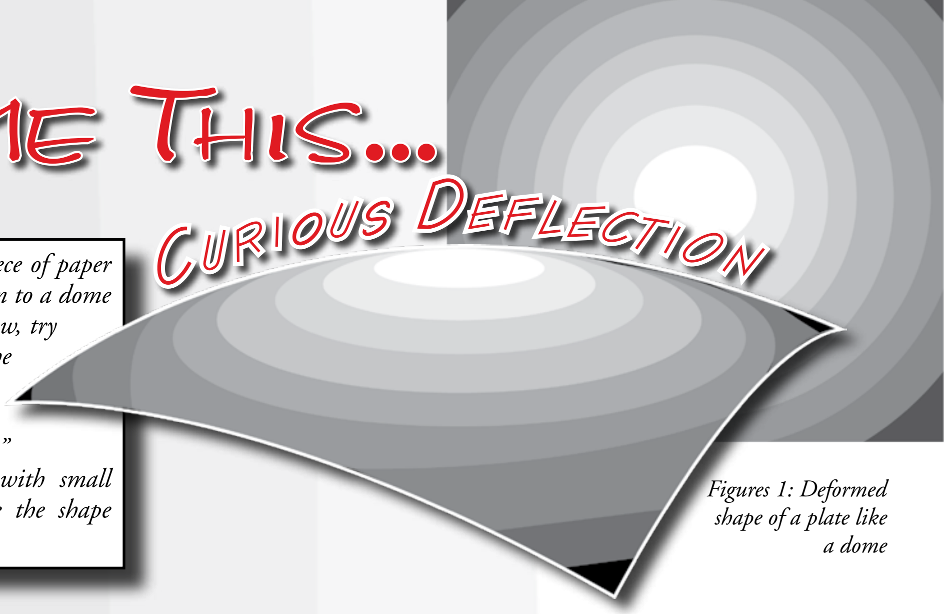
Ponder Me This...

Curious Deflection

By Erik Anders Nelson, P.E., S.E.

Imagine, if you will, the deformed shape of a piece of paper resting on your finger. Do you think it will deform to a dome (bi-directional) or an arch (uni-directional)? Now, try this simple experiment: place your finger in the middle of a piece of paper (any size or shape), and let the paper deflect. What happens? Pretty amazing, isn't it! In our untrained "mind's eye" (or using finite element, linear-elastic model with small displacement assumptions) we might anticipate the shape indicated in Figure 1, a dome-like form.



Figures 1: Deformed shape of a plate like a dome

Unfortunately our imagination and our finite element analysis are often wrong. The paper actually deforms in an arch-like unidirectional fashion as shown in *Figure 2*.

Why is this? Why does the paper "choose" to deform in a unidirectional arch, rather than in a spherical or dome-like manner?

The piece of paper can be thought of as a thin steel plate cantilevering from a center column support (for those more interested in structural engineering materials). The steel plate will behave the same as a piece of paper (ignoring fiber orientation within the paper which may force the arch to form along one axis over another).

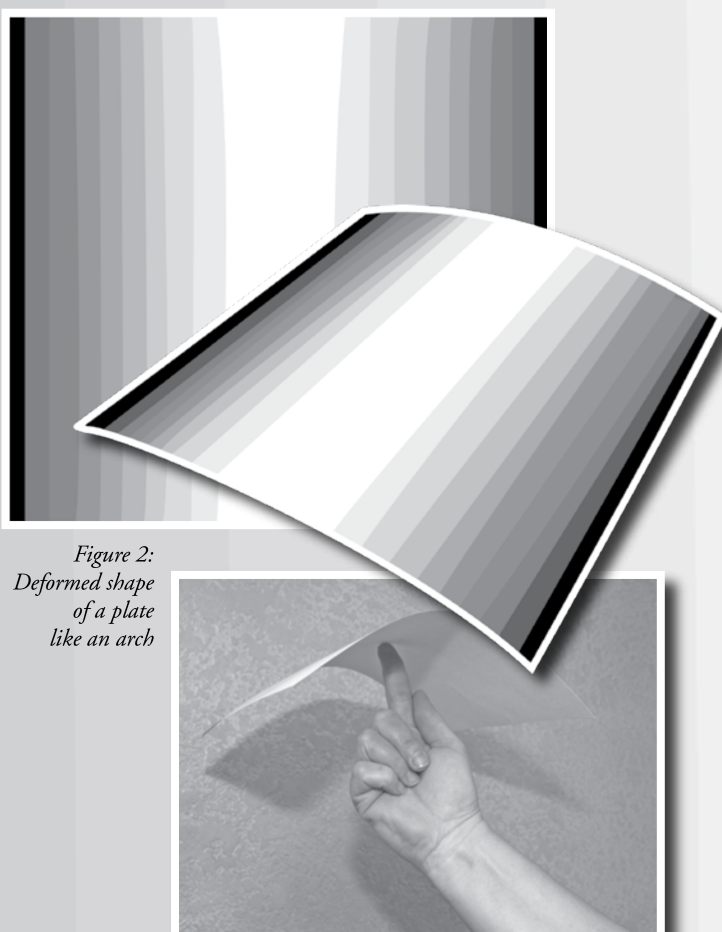


Figure 2: Deformed shape of a plate like an arch

Let us start by describing towels and umbrellas instead of paper and steel plates. If you imagine a wash towel placed over your finger, the towel hangs or drapes from your finger in tension and creases are formed to make up the loss of radial circumference (discussed more in *Figure 3*). The towel has no bending stiffness, but more importantly to our question, it has no membrane stiffness. The in-plane stiffness is critical to the paper problem. The towel is analogous to a steel plate which has yielded in a radial fashion. The radial yield lines (behaving in the plastic range) could reflect the towel draping affect, but would look more like a stiff umbrella. The umbrella deforms this way because it is made of flexible fabric.

A circle provides a better visual form to continue this discussion, and behaves in the same arch-like way. *Figure 3a* represents the un-deformed and deformed plan view of a circle acting as a dome. The inner dashed line is the circumference after deformation. Here, the circumference must decrease in length. However, the only way for the circumference to decrease in length is if the membrane is stressed in compression. If the circle did not have the option of the arch, the strain produced within the material undergoing dome-like deformation would require the shell to buckle in a wavelike pattern around the circle.

If we reverse this picture, and have a dome supported on rollers around the circumference, the membrane stress, or hoop stress near the supports, is in tension under its own weight. Assuming the material considered is paper, the final flattened shape resembles *Figure 4*. This new shape helps visualize the significance of the membrane stresses under tension and compression.

Unlike a fabric umbrella, the membrane stresses of a center supported steel plate cannot be ignored. In fact, this is why linear shell modeling gets the problem wrong; that is, the deformation of the paper (or steel plate) is too large to neglect the membrane stiffness. One would need to run a non-linear analysis to capture the membrane affect after deformation by iteration of the stiffness matrix.

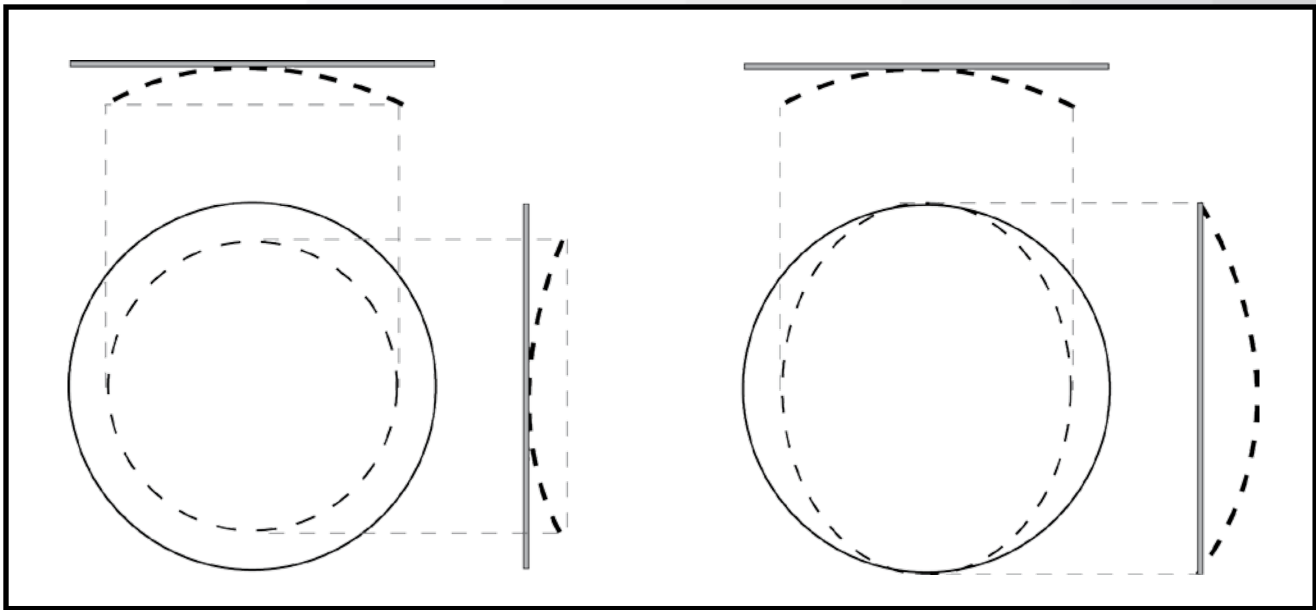


Figure 3a: Undeformed plan view of a circle acting as a dome.

Figure 3b: Deformed plan view of a circle acting as an arch.

Figure 3: A circular plate

In this case, the point supported piece of paper, has an opportunity to deform without membrane stresses, in an arch-like manner. The path chosen is not arbitrary; it is the path of least resistance. Unlike the dome, the paper has the choice to become an arch, and so it does. The circular piece of paper, if it had a conscience, is lazy. To extend this anthropomorphic metaphor further (into silliness), it simply chooses the direction which will be of minimal energy use… that is, fewer calories burned.▪

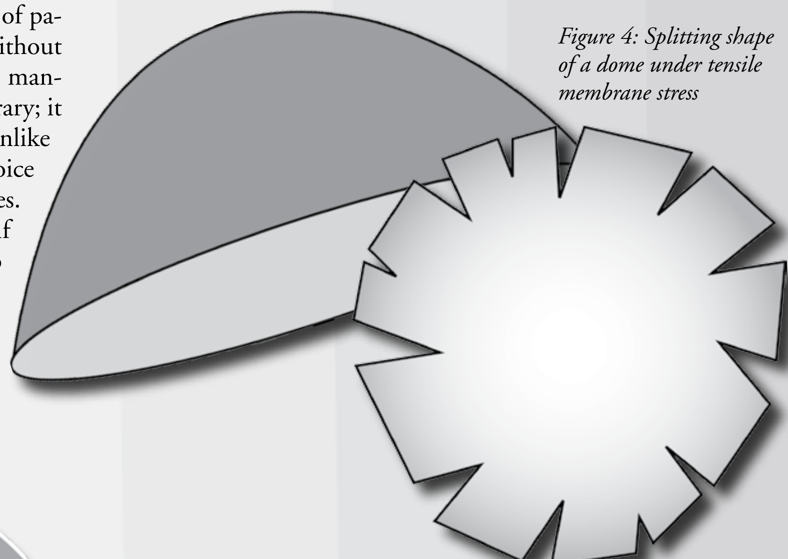


Figure 4: Splitting shape of a dome under tensile membrane stress

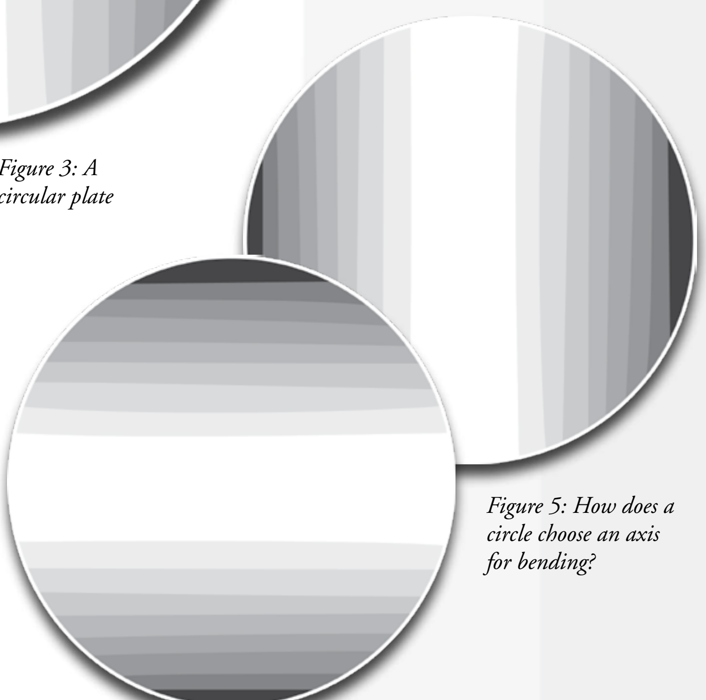


Figure 5: How does a circle choose an axis for bending?

What is possibly more intriguing, from a metaphysical point of view, is how does the circle choose an axis to bend about, 360 degrees in plan? (*Figure 5*) Have we found another Schrödinger's cat? How is the deflected paper compatible with the laws of cause and effect? It seems, prior to the deflection, the circle is in a state of indeterminacy, like the cat prior to being observed.

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