Infinite Load Path?

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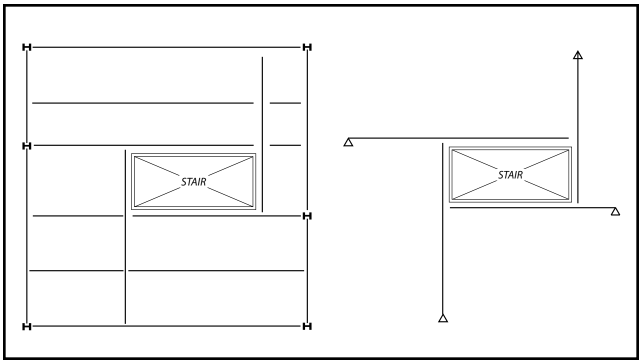
Have you ever framed out an opening with steel, and found a situation that seemed to defy common structural principles? This happened to us recently, and it wasn't the first time. We thought it worthy of an article, as peculiar framing like this can occur and sometimes is carried through construction documents and gets built! It may not be apparent at first (*Figure 1*), but when you pull out the secondary members, the problem appears (*Figure 2*).



Figures 1 (left) and 2 (right): Framing Out an Opening.

There are two associated problems here, one of analysis and the other of construction. To erect this framing layout would require temporary shoring of at least one of the inside corner connections. In practice, therefore, we should make every attempt to avoid this condition, although the completed system is perfectly stable and is easily solved by most structural analysis programs (*Figure 3*). You can build a model of it by stacking four rulers on top of each other (*Figure 4*). We will focus our attention on the more interesting problem, which is the analytical challenge.

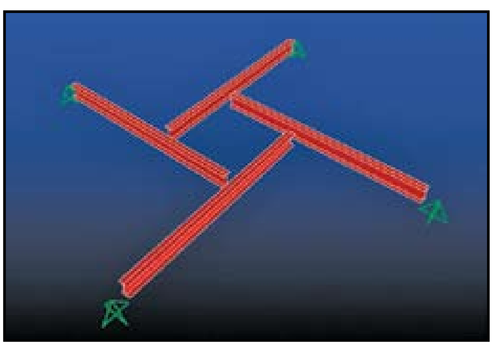


Figure 3: Analysis Model.

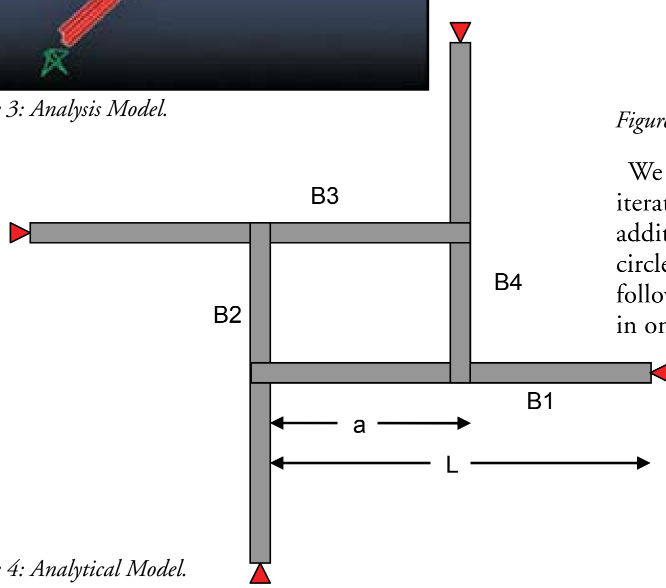


Figure 4: Analytical Model.

If we think about a uniform load applied to all four beams, how do we solve for equilibrium? And exactly how and when does the system actually reach equilibrium? There are multiple ways to solve the first problem, and the answer to the second may depend on how you solve the first. We will try three methods of solution – the iterative, the algebraic, and the direct.

Finding Equilibrium: Iterative Solution

To use the iterative solution, we first isolate one of the beams (B1), which creates a determinate structure in which each support receives half the uniform load. For the moment, we will ignore the load from the supported beam, since we don't know what the actual value of that load is yet.

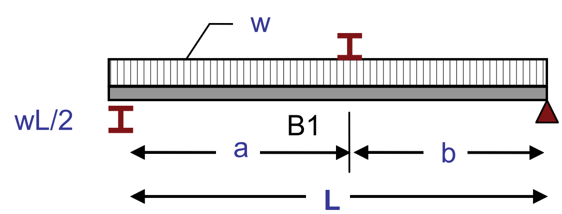


Figure 5: Iterative Solution Step A.

The next step is to analyze B2 and solve the determinate structure with the uniform load and the point load from the B1 reaction.

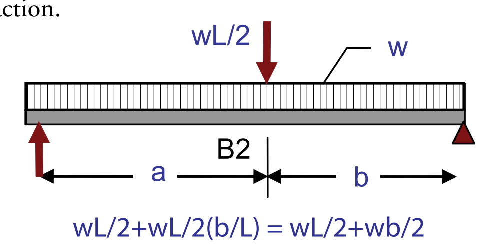


Figure 6: Iterative Solution Step B.

On to B3:

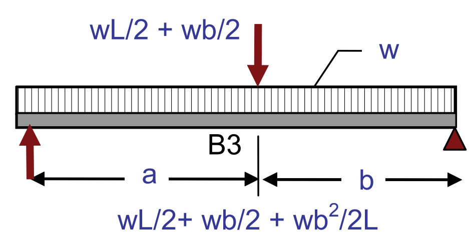


Figure 7: Iterative Solution Step C.

We can continue in this direction for any amount of iterations, but it will eventually become unnecessary as the additional load approaches zero. If we continue in circles and circles, we find the inner shears conform to a pattern. The following converging series represents the inner reaction (X) in one elegant expression:

$$X = \sum_{n=1}^{\infty} \frac{wb^{(n-1)}}{2L^{(n-2)}}$$

However, does the inner reaction actually travel in circles forever? We should not infer too much in the way of physics by this numerical solution technique, but if the numbers do somehow represent reality, then they may never converge. With this method, the reactions never ultimately equal the magnitude of the applied load. So, are the beams deflecting continuously? Is there really a finite load and an infinite load path, as a fractal has a finite area and an infinite perimeter? Before we answer that, let's look at another solution to get a second perspective.

Finding Equilibrium: Algebraic Solution

An algebraic solution is similar in that the first step is again to isolate one of the beams (B1). However, for this method, the difference is to include the load from the supported beam as a variable (X) while the beam reactions are calculated in the same order, around the inner circle. The first step is shown here:

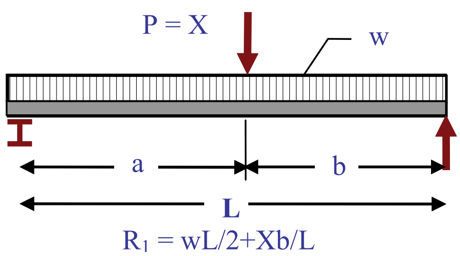


Figure 8: Algebraic Solution Step A.

The next step is to isolate B2, where the load is known in terms of X:

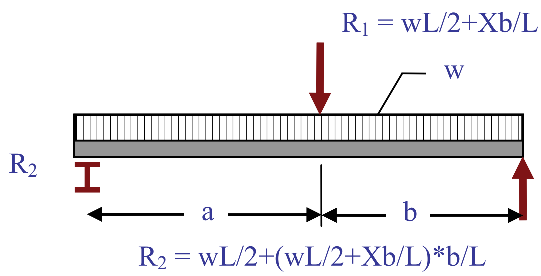


Figure 9: Algebraic Solution Step B.

Continuing around the circle to B3:

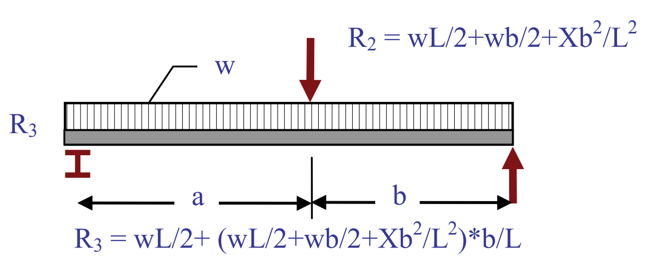


Figure 10: Algebraic Solution Step C.

One more beam (B4):

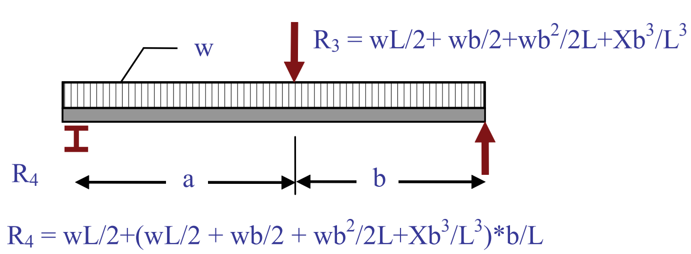


Figure 11: Algebraic Solution Step D.

The reaction from B4 (R_4) is the same force as applied to B1 in the first step. Therefore, we can substitute the equation we have just generated for R_4 into X:

$$X = wL/2+wb/2 + wb^2/2L + wb^3/2L^2+Xb^4/L^4$$

Solving for X we get:

$$X = (w/2)*(L^5+bL^4+b^2L^3+b^3L^2)/(L^4-b^4)$$

This expression can be simplified algebraically to:

$$X = wL^2/2a$$

Now, we can also solve for all of the outside reactions and set them equal to the total applied load. The free body diagram looks like this:

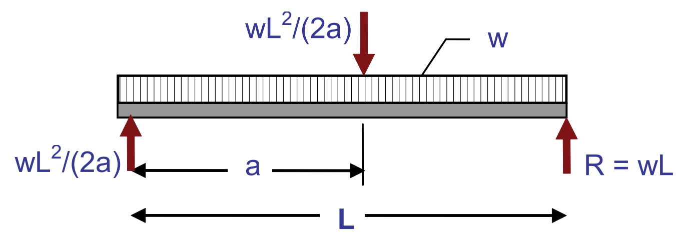


Figure 12: Algebraic Solution.

So in this method, we obtain a closed-form solution. The inner reaction is $wL^2/(2a)$ and not an infinite, converging series. Therefore, this method looks very different from the answer that we found with the iterative approach, even though both solutions produce mathematically equivalent expressions. It is worth mentioning again that we should not infer too much in the way of physics by this numerical solution technique either. Let's look at one more method.

Finding Equilibrium: Direct Solution

There is third method which is quite simple, using static principles that you can do in your head. If we know the total load, then the reaction is equal to this total load divided by four supports (assuming that the structure is symmetric). If the reaction is equal to the uniform load on the beam, we know that an inner couple is formed (equal and opposite). We will set each force of the couple equal to 'X'.

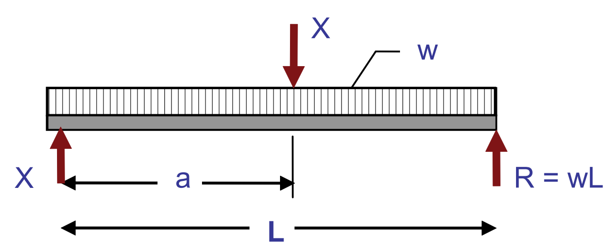


Figure 13: Direct Solution.

Now we can sum moments about the supports and we find…

$\sum M = 0$ (at end reaction on right side)
$XL - X(L-a) - wL(L/2) = 0$
$XL - XL + Xa - wL^2/2 = 0$
$X = wL^2/(2a)$ (Same solution as the algebraic one)

continued on next page

One interesting thing to note about the value of X (for all the solution methods) is that as the opening gets smaller and 'a' approaches 0, X dramatically increases. For example, if a = L/100:

$$X = wL^2/(2a) = wL^2/(0.02L)$$
$$X = 50wL$$

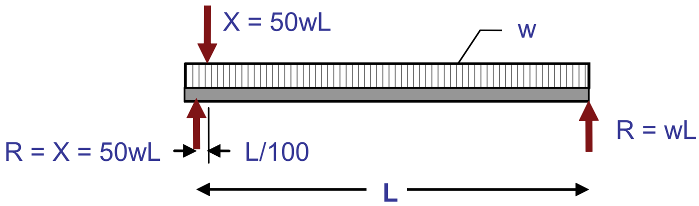


Figure 14: Large Shear if Opening is Small.

The shear in the support next to the load is also equal to this value. This large force would really be something to watch out for when designing members and connections in the unlikely event that a very small opening was to be framed in this manner. Note that, if 'a' equals 0, the system is unstable, at least in the linear geometry range.

How and When Does It Converge?

Depending on what solution makes more sense, you may answer this question differently than another engineer. If you built the structure out of rulers, then you probably noticed that as soon as any load is applied, the structure deflects instantaneously, at least as far as the human eye can tell. Although the numerical solution techniques above may not represent what is actually happening in the system over time, we can still look at the speed of the shear stress waves to gain more information.

If you accept the conventional view that equilibrium actually occurs, it is logical to say that the total reactions are felt almost instantaneously. The shear wave travels (one time) from the load, at midspan of the beam, to the support. Shear waves in steel travel at about 10,000 feet per second. Assuming that we have a 20-foot beam, and the reaction travels half that distance, then the load reaches the supports in about 0.001 seconds.

If instead you assume that reality is more consistent with the iterative method, then maybe it takes more time as some of the load must travel around the inner circle. Instead of the total load traveling to the support in 0.001 seconds, the load increases at each step of the iterative method (Figure 15).

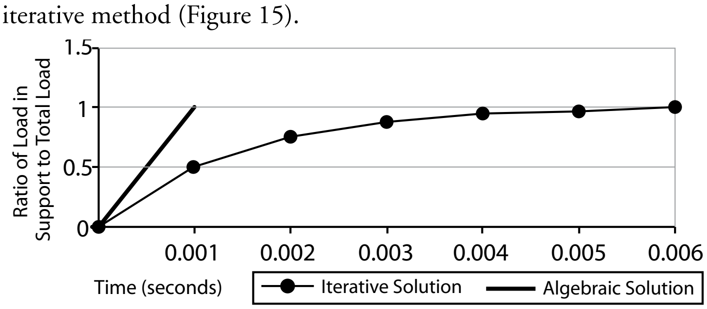


Figure 15: How quickly does the load travel?

Even if we exaggerate the structure by making 'a' very small, the reaction converges at about the same rate. Try it out and see.

Unfortunately, it appears that a 'kitchen table' experiment will not provide the answer. We know the arrangement works in practice as firefighters have been using this technique for creating seats from four interlocking arms for many years. But does this system actually reach equilibrium? Before we answer that, we need to look at one more property of the structure.

Dynamics

Thus far, we have mostly neglected an important issue, the dynamics of the system. Since we have brought time into the picture, dynamics should be included.

Anytime we load a structure – in this case, when we remove the shoring – the structure will deflect. Deflection implies movement, which involves time and momentum. The momentum of the load, in turn, will generate deflections greater than the equilibrium deflection (*Figure 16*), oscillating about the equilibrium deflection. And since the reactions are a function of the deflection (strain causes stress), then the reactions will actually be greater than the applied load, unless the system is over-damped, for half of the duration of the oscillations:

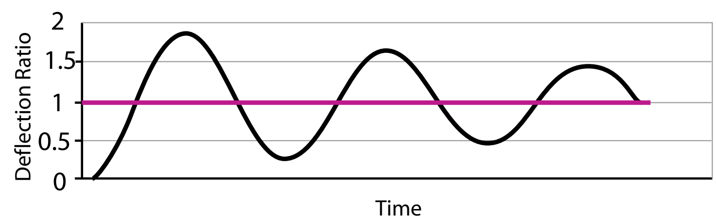


Figure 16: Deflection over Time.

Therefore, in all structures, the reactions are rarely exactly equal to the applied load. Even after the structure has been loaded, and any damping has taken effect, the structure will still ever so slightly be oscillating, changing the reactions in the supports from the vibrations all around the structure that are caused by people walking close by, mechanical equipment running, etc.

The system of stacked beams is no different from a simply supported beam in this respect. The reactions are rarely exactly equal to the applied load. Does this affect the answer to the question: How and when does the system converge?

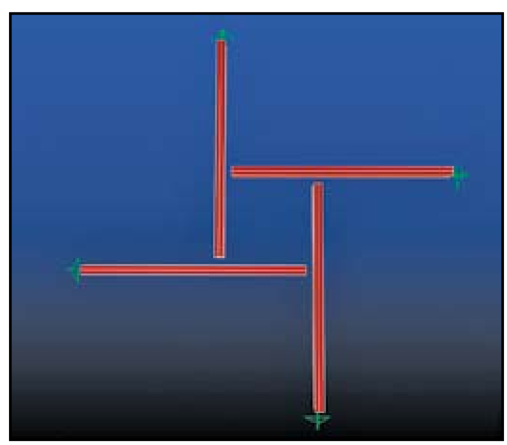


Figure 17: How fast the does this system reach equilibrium?

Conceptually, we would agree that the inner reaction keeps traveling, ad infinitum, such that the system can be defined as an active system. Since each beam is its own determinate structure, the load continues to circulate (how else could it behave?), and the intermolecular bonds continue to extend and compress ever so slightly. In this sense, we suppose, it is continuously deforming in ever-smaller increments and never actually reaches equilibrium. A living fractal!

It should be noted that shear waves travel much faster than the period of the structure. Because this is the case and the shear waves travel almost instantaneously, then one could argue that the system is virtually static. Either way, it is perfectly stable.

"Infinite Load Path" beam systems can be shown to work for larger-scale structures, such as domes made out of stacked popsicle sticks (*Figure 18*).

As for the opening around the stair, we are going to keep this framing layout to see how the unsuspecting contractor feels about the construction. For the first time, we look forward to the RFIs!▪

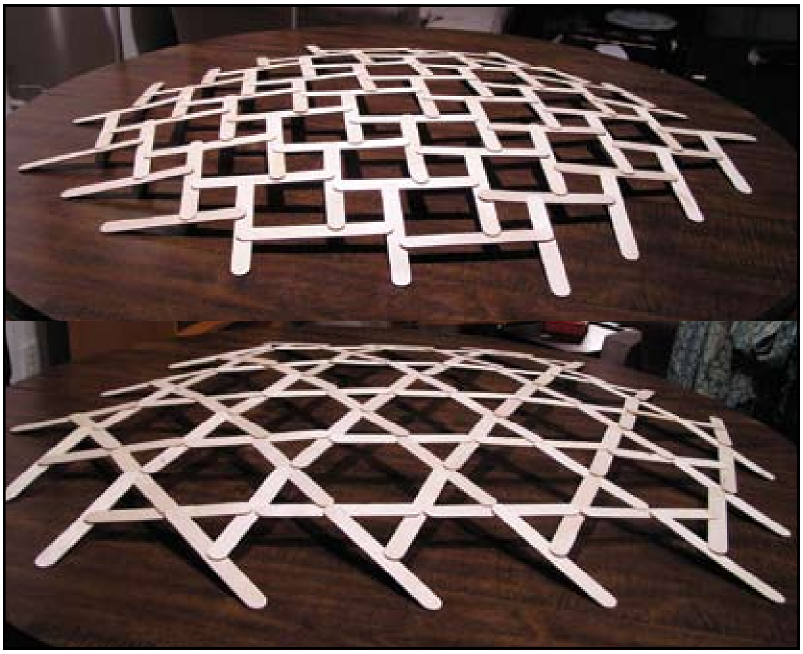


Figure 18: Popsicle Domes of Stacked Members.

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