Post-Yield Stiffness Effects on Moment Redistribution in Continuous Reinforced Concrete Beams

By Pedro Silva, Ph.D., P.E.

Reinforced concrete (RC) beams of the type shown in *Figure 1* are commonly designed using moment redistribution principles. RC continuous beams or plane frames may have any number of spans or boundary restraints; the work presented in this article is for a simply supported, two-span, continuous RC beam, but many of the conclusions can be extrapolated to other situations. In design, these continuous members are typically assumed to display an elasto-plastic response, which means that after yielding of the tension steel any increase in stiffness due to strain hardening is neglected. In reality, beams subjected to large inelastic strain levels may attain a significant post-yield stiffness, which has a strong effect on the moment redistribution of continuous RC beams.

In this article, the basics of moment redistribution are discussed as a function of curvature ductility capacity using finite element subroutines. The author further illustrates the principles of moment redistribution in the design of a two-span RC continuous beam, including the potential effects of post-yield stiffness.

Basics of Moment Redistribution

Sections 8.4.1 and 8.4.3 of ACI 318-05 state that the level of moment redistribution (MR) that is permitted in a continuous RC beam is:

$$7.5\% \leq MR = 1000\varepsilon_t \leq 20\% \qquad \text{Equation 1}$$

where ε_t is the level of strain in the extreme tension reinforcement. As such, this strain must be at least 0.0075 before MR is permitted. The permissible levels of MR defined by *Equation 1* are conservative, and results derived from this study show that strain levels will in many cases fall significantly below 0.005, which violates the ACI 318-05 limit for a tension-controlled design. Stipulated by *Equation 1*, the amounts of MR that can be allowed in the design of continuous RC beams are only expressed as a function of tensile strains. Because of its generality, the work presented in this article will evaluate MR in RC structures as a function of curvature ductility capacity. Previously the author has derived an expression to obtain the curvature ductility, μ_ϕ, as a function of tensile strain. The relationships of MR in terms of tensile strain and curvature ductility are outlined in *Figure 2*.

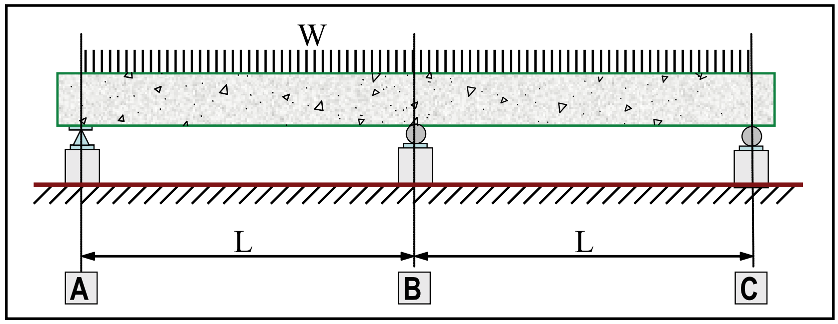

Figure 1: Two-span continuous beam under uniform loads.

Formulation of MR as a function of curvature ductility capacity is presented in terms of the moment curvature (M-ϕ) relationships and the statically indeterminate beam shown in *Figure 3*, which is a simplified version for the analysis of the two-span beam shown in *Figure 1*. The beam is uniformly loaded and is pinned and fixed at ends A and B, respectively. Under an increasing load, the beam will deform elastically up to yielding and then plastically at end B.

For the nonlinear part of the analysis, two released structures may be considered. For Release 1 the beam is considered perfectly plastic at end B, and in Release 2 the beam can be considered restrained by a plastic rotational spring with the stiffness, β, idealized in terms of the post-yield stiffness, r; initial stiffness, EI; and plastic hinge length as a function of beam span length, λL.

Modeling the inelastic response of the beam in terms of Release 1 follows the elasto-plastic idealization presented in *Figure 3(a)*. The beam is assumed to begin deforming plastically at end B when the moment and curvature reach M_n and ϕ_y, respectively. After this stage, the incremental uniform applied load on the beam will impose inelastic rotations and curvatures at support B. The amount of MR that the beam can sustain is computed as follows:

$$MR = 1 - \frac{1}{1 + 3\lambda\left(\mu_\varphi - 1\right)} \qquad \text{Equation 2}$$

Modeling the inelastic response of the beam in terms of Release 2 follows the bilinear M-ϕ relationship presented in *Figure 3(a)*. The beam begins to deform plastically at end B when the moment and curvature reach M_y, and ϕ_y, respectively. After this stage, the beam develops plastic rotations and curvatures that include the post-yield stiffness and plastic hinge length. Following similar steps, in Release 2 the amount of MR that the beam can sustain is computed as follows:

$$MR = 1 - \frac{1}{1 + \left(3\lambda + r\right)\left(\mu_\varphi - 1\right)} \qquad \text{Equation 3}$$

Equations 2 and *3* can be used to compute the amount of MR that a beam can sustain as a function of the plastic hinge length, post-yield stiffness and curvature

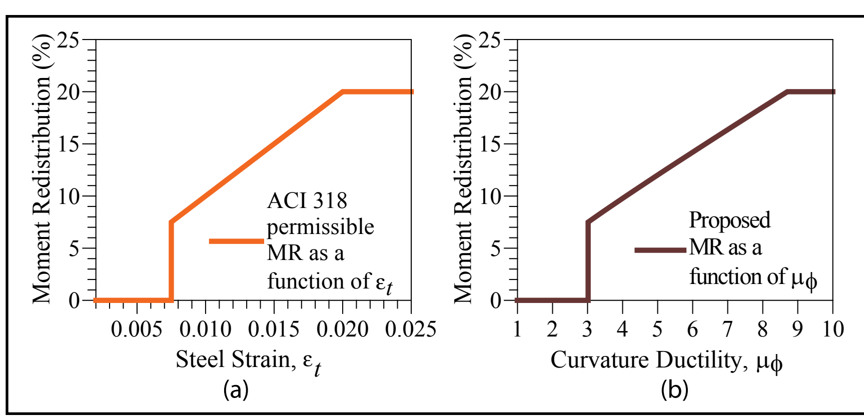

Figure 2: Moment redistribution (a) Function of ε_t, (b) Function of μ_ϕ.

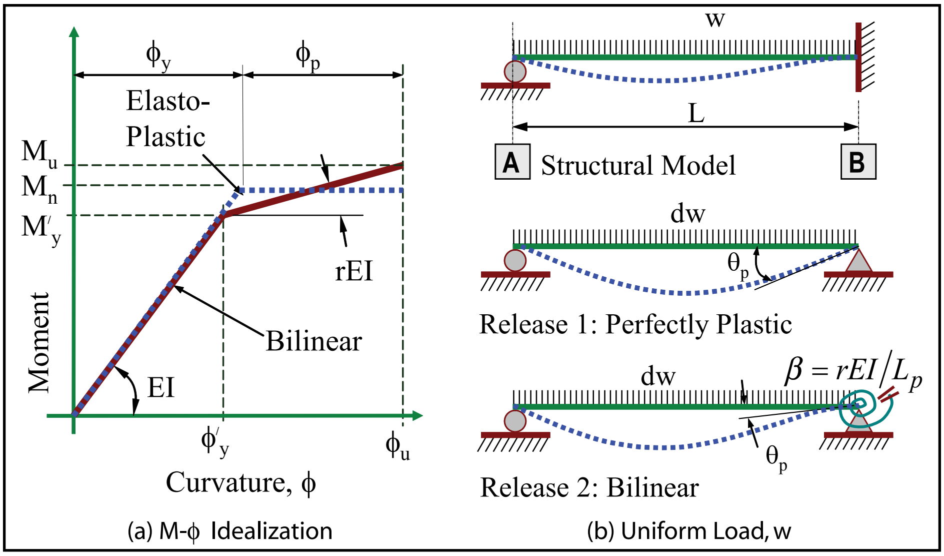

Figure 3: Basics of moment redistribution.

ductility capacity. Obviously, these principles of MR capacity only apply to the beam geometry presented in *Figure 1*.

The permissible levels of MR in two-span continuous beams that correspond to the two releases are depicted graphically in *Figure 4*. The post-yield stiffness (r) and plastic hinge length (λL) have a marked effect on the MR capacity of two-span continuous beams. It is envisioned that this same observation will also apply to other continuous structures.

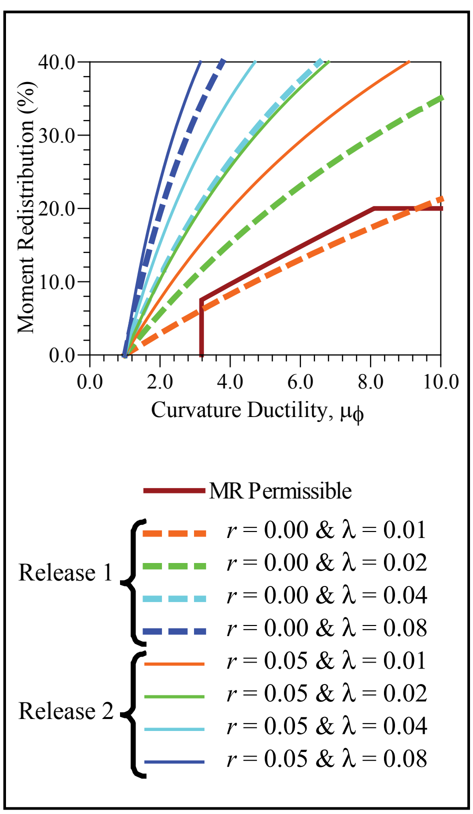

Figure 4: Moment redistribution versus ductility.

Some other trends of the MR levels presented in *Figure 4* are as follows: (i) as the post-yield stiffness ratio increases, so does MR; (ii) as the plastic hinge length increases, so does MR; (iii) the curve for r=0.00 and λ=0.01 follows below the permissible MR curve computed based on *Equation 1*, and depicted in *Figure 2*. The next section presents the effects that these trends have on the actual performance of beams designed using MR principles.

Design and Performance Evaluation

As discussed, the levels of MR that can be achieved in continuous beams depend strictly on the plastic rotation capacity of members at plastic hinges. In this section, a design example has been established to investigate the effects that post-yield stiffness has on MR.

Reflecting the parameters of *Table 1*, design required a beam with the cross-section dimensions and reinforcement layout shown in *Figure 5(a)*, which consists of 6-#5 (Grade 60) top and bottom bars. The moment-curvature analysis for this section is presented in *Figure 5(b)*. The solid curve is the moment-curvature section analysis that is used to evaluate the performance of the two-span continuous RC beam under Release 2 with r=0.035. The dashed curve is for the same evaluation under Release 1 with r=0. It is important to emphasize that in current practice, r=0 is generally assumed for design.

From the moment-curvature analysis, the curvature ductility capacity of the section is nearly $\mu_\phi \approx 6.5$. From *Figure 2(b)* this ductility capacity translates into a MR capacity of 15.7%. Comparatively, for Release 1 the MR that the section can develop is 12.1%. On the other hand, for Release 2 the two-span beam can now develop a much greater MR=61.2%. This simple example clearly shows that the actual MR that the beam can develop is significantly higher than what is allowed by ACI 318. *Figure 6 (page 20)* shows the moment profiles for three cases. One curve shows the profiles considering the elastic design condition, another corresponds to r=0, and the third represents r=0.035. It is not apparent from these curves the salient differences between a design that considers r=0 and the actual response of the beam with r=0.035.

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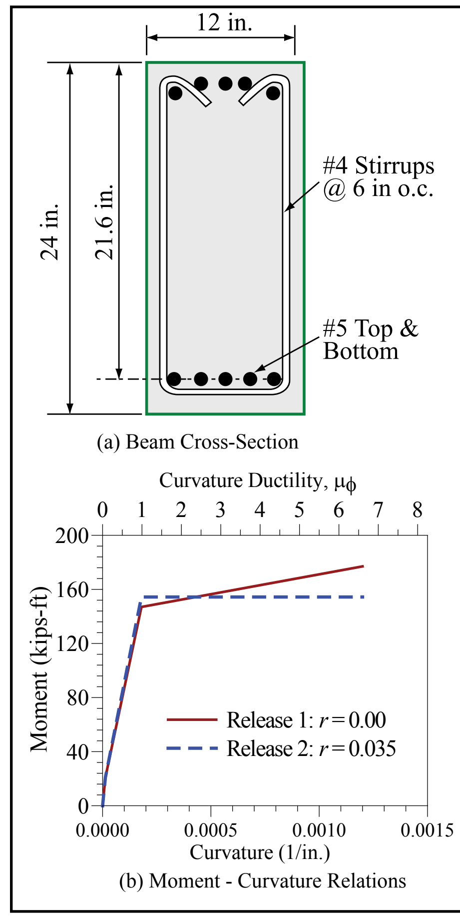

Figure 5: Cross Section Dimensions and Capacity Analysis.

Span length = 20 feet	Steel bars required = 6-#5 (Grade 60)
Uniform dead load = 900 plf	Uniform Live Load = 1400 plf
Dead load factor = 1.2	Live load factor = 1.6
Ultimate factored load = 3,320 plf	MR per ACI (6-#5) = 15.7% for μ_ϕ
Release 1: With r=0, MR = 12.1%	Release 2: With r=0.035, MR = 61.2%

Table 1: Design Parameters.

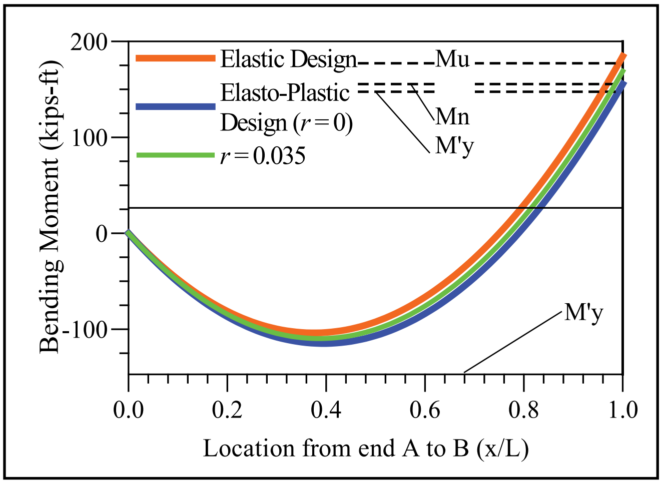

Figure 6: Moment profiles for end spans A-B.

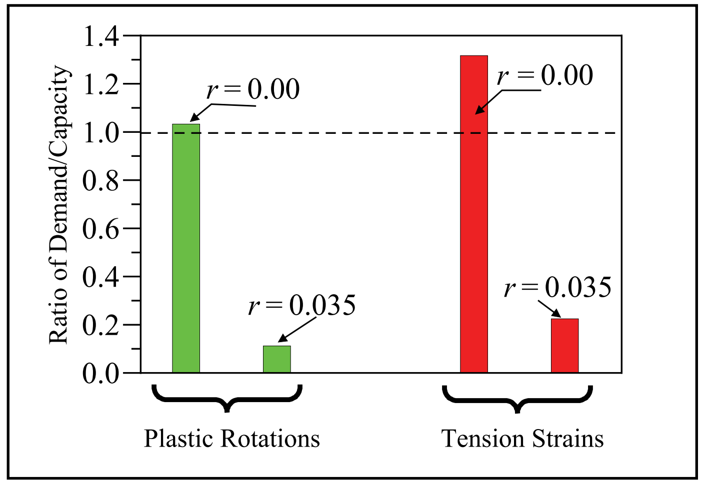

Figure 7: Ratio of Demand versus Capacity.

Figure 7 shows the ratio of the plastic rotation and tensile strain demand versus capacity. For r=0, demand exceeds the section plastic rotation and tensile strain capacity by a ratio of 1.02 and 1.3, respectively. For r=0.035, there is a drastic decrease in the demand versus capacity ratio to 0.10 and 0.25, indicating that the degree of conservatism is on the order of 10. These ratios show that post-yield stiffness has a marked effect on the moment redistribution of continuous RC beams.

Future Investigations

This article presented some of the basics of moment redistribution principles and applied them to a two-span continuous RC beam. Results show that post-yield stiffness has a marked effect, an important observation that should be investigated in further detail for structures that have a higher order of indeterminacy. Issues of moment redistribution for continuous beams with a number of spans greater than two and plane frames will be undertaken in the future.▪

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