Optimum Beam-to-Column Stiffness Ratio of Portal Frames under Lateral Loads

By Pedro Silva, Ph.D., P.E. and Sameh S. Badie, Ph.D., P.E.

An article in the March 2008 issue of STRUCTURE® magazine discussed design optimization of portal frames, such as those shown in *Figure 1*, under gravity loads. This follow-up article discusses similar design optimization under the influence of lateral loads. It presents, both analytically and graphically, the procedure that is used to establish the design optimization. Since the drift ratio, defined as the ratio of the frame lateral deflection to column height, δ, is one of the controlling factors in the design of portal frames subjected to lateral loads, the design optimization is performed as a function thereof. Design optimization is achieved by minimizing the sum of normalized column and beam moments of inertia necessary to satisfy a prescribed drift ratio limit. The authors performed their analyses using finite element subroutines implemented in Matlab®. Following the results, the authors present simple graphical procedures that can be used to illustrate the optimum beam-to-column stiffness (I_b/I_c) ratios in the design of portal frames under lateral loads.

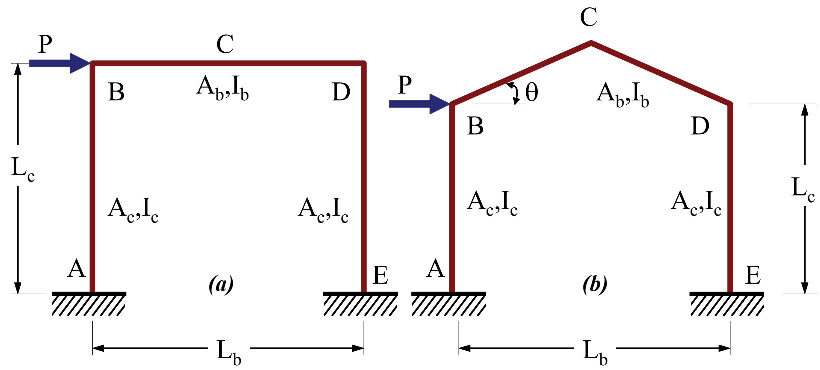


Figure 1: Portal frames under lateral loads (a) Flat frame, (b) Pitched frame.

Effect on the Frame Behavior

Under lateral loads, frame design is highly dependent on the beam-to-column stiffness ratio, beam-span-to-column-height ratio, and the column end supports.

Fixed Support Conditions

If axial deformations are ignored, using slope-deflection methods it can be demonstrated that for a flat frame with fixed supports, as shown in *Figure 1(a)*, the lateral deflection, Δ_f, is:

$$\Delta_f = \left(\frac{6\alpha+4\kappa}{6\alpha+\kappa}\right)\frac{PL_c^3}{24EI_c} \le \delta L_c : \quad \text{Fixed Support}$$

Equation (1)

In *Equation (1)*, α is the beam-to-column moment of inertia ratio, given by $\alpha = I_b/I_c$, and δ is the drift ratio designated as the ratio of lateral deflection to column height. This equation also includes an arbitrary beam-span-to-column-height ratio, designated herein as $\kappa = L_b/L_c$.

The authors used this equation to develop *Table 1*, depicting the frame deflection, Δ_f, under three beam-to-column ratios, $\alpha = I_b/I_c$, and for a beam-span-to-column-height ratio of κ=1. For α=0 (corresponding to infinitely stiff columns), the frame deflection, Δ_f, is four times higher than the deflection for α=∞ (corresponding to infinitely stiff beams). This is the case because for α=0 the frame response reverts to that of two cantilever columns, and for α=∞ it reverts to that of two propped cantilever columns. The results for α=0.74 are also shown in *Table 1*.

In order to meet the drift ratio, δ, the column stiffness is obtained by solving for $\Delta_f \le \delta L_c$ in *Equation (1)*. Normalizing the required column stiffness in terms of the variable EI_c/PL_c^2 gives the following result:

$$\lambda_c = \left(\frac{6\alpha+4\kappa}{6\alpha+\kappa}\right)\frac{1}{24\delta} : \quad \text{Fixed Support}$$

Equation (2)

In *Table 1*, the normalized column stiffness is given in terms of the variable λ, showing once again that the required λ_c for α=0 is 4 times higher than for α=∞.

The authors developed expressions similar to those presented in *Table 1* for other values of roof pitch, beam-to-column I's, α, and κ. However, for brevity and illustrative purposes, only the results for κ=1.5 are presented in *Figure 2(a)*. In this figure is shown the ratio λ_c as a function of α for different roof pitches. It is clear that as a function of α, λ_c is insensitive to variations in the roof pitch. This figure also shows that as α increases, λ_c reduces, which matches the results discussed in *Table 1*.

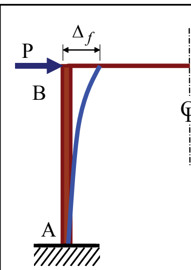
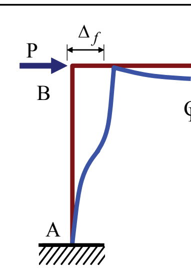
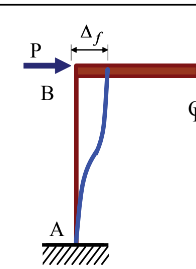
The design optimization rule can be obtained in terms of summing the normalized ratios $\alpha+\lambda_c$. Once again, for κ=1.5, the sum $\lambda_c+\alpha$ as a function of α is shown in *Figure 2(b)*. Curves in this figure clearly indicate that a minimum point exists; this point corresponds to the optimum beam-to-column stiffness ratio, α. Curves in this figure also show that the minimum point is nearly insensitive to the roof pitch.

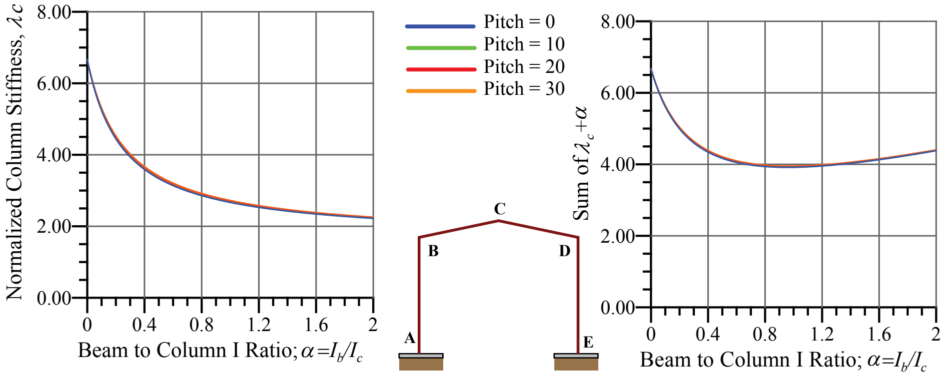
Consequently, the optimum beam-to-column stiffness ratio, α, can be expressed in terms of the following minimization rule:

$$\alpha^* = \left[\frac{d}{d\alpha}[\lambda_c+\alpha]=0\right] \Rightarrow \alpha^* = \left(\sqrt{\frac{3\kappa}{4\delta}}-\kappa\right)\frac{1}{6} : \quad \text{Fixed Support}$$

Equation (3)

Table 1: Deflected shape for varying values of α = I_b/I_c for κ = 1.00: Fixed frame.

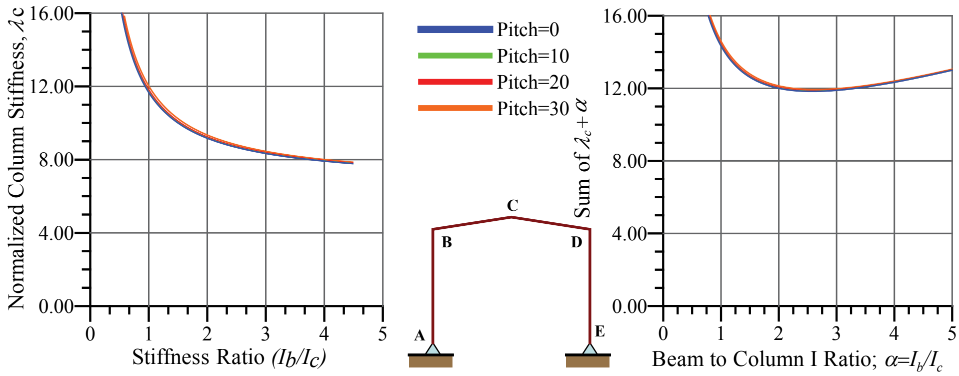
	(a) α = I_b/I_c = zero	(b) α = I_b/I_c = 0.74	(c) α = I_b/I_c = infinity (∞)
Deflected shape			
Δ_f	$+\frac{4PL_c^3}{24EI_c}$	$+\frac{1.5515\times PL_c^3}{24EI_c}$	$+\frac{PL_c^3}{24EI_c}$
λ_c	$+\frac{4}{24\delta}$	$+\frac{1.5515}{24\delta}$	$+\frac{1}{24\delta}$



(a) κ (L_b/L_c) =1.5, Fixed support

(b) κ (L_b/L_c) =1.5, Fixed support

Figure 2: Results for fixed supports.

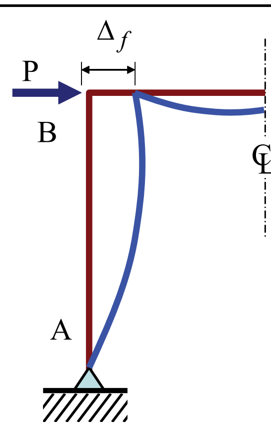
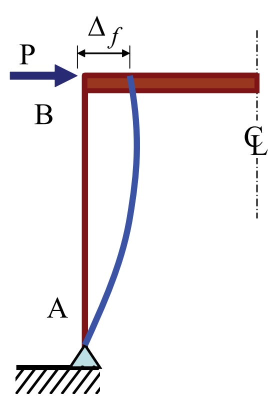


(a) κ (L_b/L_c) =1.5, Pinned support

(b) κ (L_b/L_c) =1.5, Pinned support

Figure 3: Results for pinned supports.

Table 2: Deflected shape for varying values of $\alpha=I_b/I_c$ for $\kappa=1.00$: Pinned frame.

	(b) I_b/I_c = 1.8257	(c) I_b/I_c = infinity
Deflected shape		
Δ_f	$+\dfrac{5.0955 \times PL_c^3}{24EI_c}$	$\dfrac{4PL_c^3}{24EI_c}$
λ_c	$+\dfrac{5.0955}{24\delta}$	$+\dfrac{1}{24\delta}$

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Pinned support conditions

Following the same methodology, the authors developed expressions for frames with pinned supports. As before, if axial deformations are ignored, for a flat frame with pinned supports, the lateral deflection, Δ_f, is given by:

$$\Delta_f = \left(4+\frac{2\kappa}{\alpha}\right)\frac{PL_c^3}{24EI_c} \le \delta L_c : \quad \text{Pinned Support}$$

Equation (4)

The authors used *Equation (4)* to develop *Table 2*, depicting the frame deflection, Δ_f, under α=1.8257 and infinity, and for an arbitrary κ. Clearly, in the condition of α=0 the frame is unstable; consequently, this case is not shown. In order to meet the required drift ratio, δ, the required column stiffness is obtained by solving for $\Delta_f \le \delta L$, as depicted in *Equation (4)*. Normalizing the required column stiffness in terms of EI_c/PL_c^2 results in *Equation (5)*.

As before, the authors developed expressions similar to those presented in *Table 2* for other conditions and for brevity and illustrative purposes, only the results for κ=1.5 are presented in *Figure 3(a)*.

$$\lambda_c = \left(4+\frac{2\kappa}{\alpha}\right)\frac{1}{24\delta} : \quad \text{Pinned Support}$$

Equation (5)

For a frame with pinned supports, *Figure 3(b)* shows once again that there is an optimum value for design, corresponding to the minimum point on these curves. The optimization rule is mathematically expressed by *Equation (6)*.

$$\alpha^* = \left[\frac{d}{d\alpha}[\lambda_c+\alpha]=0\right] \Rightarrow \alpha^* = \frac{1}{6}\sqrt{\frac{3\kappa}{\delta}} : \quad \text{Pinned Support}$$

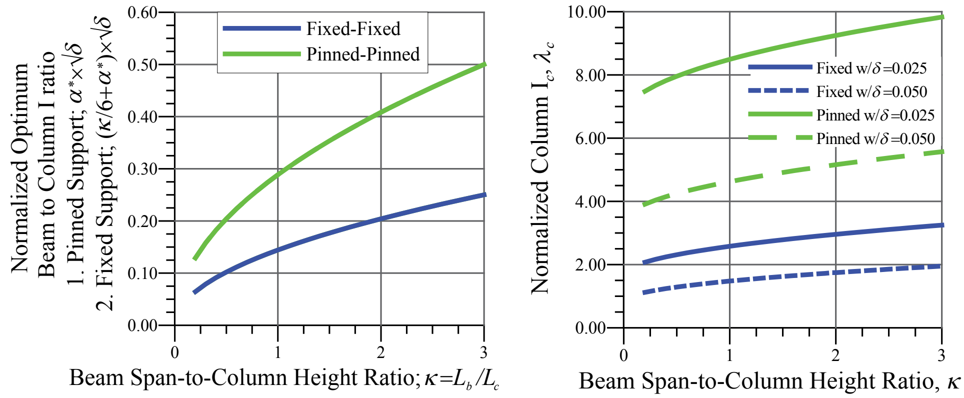
Equation (6)

Design Optimization

Graphical approach

For frames with fixed and pinned supports, the results of normalizing *Equations (3)* and *(6)* in terms of $(\alpha^*+\kappa/6)\sqrt{\delta}$ and $\alpha^*\sqrt{\delta}$, respectively, are presented in *Figure 4(a)*. The results of substituting *Equation (3)* and *Equation (6)* into *Equations (2)* and *(5)* are presented in *Figure 4(b)*. *Figure 4(a)* and the work developed in this article show that:

1) As $\kappa = L_b/L_c$ increases, so does the optimum stiffness ratio, α^*.
2) In frames with pinned supports, the optimum stiffness ratio is significantly higher than for a frame with fixed supports.
3) As the drift ratio increases, so does the required stiffness ratio.



(a) Normalized $(\alpha^*+\kappa/6)\sqrt{\delta}$ and $\alpha^*\sqrt{\delta}$ *(b)* Normalized column stiffness, λ_c

Figure 4: Optimum ratio charts for lateral loads.

Table 3: Design example results for a roof pitch of 20°.

Supports (1)	Using Figure 4(a) (2)	α^* (3)	λ_c a) (4)	$I_c=\lambda_c\frac{PL_c^2}{E_s}$ cm^4 (in^4) (5)	$I_b=\alpha^* I_c$ mm^4 (in^4) (6)	Δ_f b) mm (in.) (7)
Fixed	$(\alpha^*+\kappa/6)\sqrt{\delta}$=0.177	0.868	2.785	1.66 (398)	1.44 (346)	92.02 (3.62)
Pinned	$\alpha^*\sqrt{\delta}=0.354$	2.236	8.903	5.30 (1,273)	11.85 (2,846)	

a) Using Figure 4 (b) b) Using Equations (1) and (4)

Design example using optimization rules

The authors used the charts in *Figure 4(b)* to develop the design example presented in this section. Design optimization follows the design calculations presented in *Table 3*. In this section, the frame has a roof pitch of 20° and consists of Grade 50 steel with E_s=200 GPa (29,000ksi), an applied lateral load of 889.6 kN (200 kips), and beam and column lengths of L_b=5.49 meters (18 feet) and L_c= 3.66 meters (12 feet), providing an aspect ratio κ=1.50. Using a design drift ratio δ=0.025 (1/40), the limiting lateral deflection of the frame was $\Delta_f \le \delta L_c$ = 91.44 millimeters (3.6 inches). The computed values, shown in *Table 3* column (7), only slightly exceed this value, indicating that the design charts presented in *Figure 4* can be used reliably to compute the optimum moments of inertia I_c and I_b.

Future investigations

The design charts developed herein considered only frames under lateral loads. The design optimizations based only on drift ratios may not necessarily apply to cases where gravity loads also contribute significantly to the frame response. In future investigations, design optimizations will consider the frame response dominated by both gravity and lateral loads.■

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